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Involutive automorphisms of the class of affine Kac–Moody algebras $\mathrm{B}_\ell^{(1)}$

S P Clarke and J F Cornwell

School of Physics and Astronomy, University of St Andrews, North Haugh, St Andrews, Fife, KY16 9SS, UK

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Abstract. All the conjugacy classes of involutive automorphisms of the affine Kac–Moody algebras $B_\ell^{(1)}$ for $l \geqslant 1$ are determined using the matrix formulation of automorphisms of an affine Kac–Moody algebra.

1. Introduction

1.1. Preliminaries

In a previous paper (Cornwell [1]), the important role played by the automorphism groups of an affine Kac–Moody algebra was discussed, and a matrix formulation was developed for dealing with general automorphisms of affine untwisted Kac–Moody algebras. Subsequently in Cornwell [2–4], this method was used to investigate the involutive automorphisms of the algebras $\mathrm{A}_\ell^{(1)}$, ℓ taking all positive integer values. In this paper this analysis is continued for the algebras $\mathrm{B}_\ell^{(1)}$ for all integer values of ℓ. Here we will retain the notations and conventions used in those papers (Cornwell [1–4] which will be referred to as papers I–IV respectively), with the convention that (I.6) refers to the equation numbered (6) in paper I, and so on, whereas (6) is the sixth labelled equation of the present paper.

In addition to the notation already developed, it is helpful to introduce some new notations, conventions and terminology, for ease of understanding, as well as for conciseness.

(1) We recall that in the matrix formulation, use was made of matrices $\mathbf{U}(t)$, where a typical $\mathbf{U}(t)$ is such that $\mathbf{U}(t)$ is invertible, and both $\mathbf{U}(t)$ and its inverse are composed entirely of Laurent polynomials. Such a matrix will subsequently be referred to as a'Laurent polynomial matrix' or simply as a 'Laurent matrix'.

(2) As an analogy to the notation 'diag$\{a, \ldots, z\}$', we define the expression 'offdiag' for the minor diagonal case. For example,

$$\mathrm{offdiag}\{a, b, c\} = \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix} .$$

Thus, in the 'offdiag' term we have defined, the first component is to be interpreted as the top right entry of the matrix.

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(3) Similarly, 'dsum' indicates a direct sum, so, for example,

$$\mathrm{dsum}\{\mathbf{a},\mathbf{b},\ldots,\mathbf{y},\mathbf{z}\}=\begin{pmatrix}\mathbf{a} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{b} & \cdots & \mathbf{0} & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{y} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{z}\end{pmatrix}$$

where **a**,**b**,**y**,**z** are all square submatrices.

(4) We define $(2\ell+1)\times(2\ell+1)$ matrices $\mathbf{X}_j$ and $\mathbf{e}_{j,k}$ where j,k take integer values from $1,\ldots,\ell$. $\mathbf{X}_j$ has just two non-zero entries, the jth diagonal entry which is 1 and the $(2\ell+2-j)$th diagonal entry which is -1. $\mathbf{e}_{j,k}$ has just one non-zero entry (the (j,k)th) which is 1.

1.2. The Kac–Moody algebra $B_\ell^{(1)}$

In this paper we are concerned ultimately with the Kac–Moody algebra $\mathrm{B}_\ell^{(1)}$. In particular, we aim to use the matrix formulation to obtain conjugacy class representatives from each class of involutive automorphisms. The generalized Cartan matrix for $\mathrm{B}_\ell^{(1)}$is the $(\ell+1)\mathrm{x}(\ell+1)$ matrix **A**, where

$$\mathbf{A}=\begin{pmatrix}2 & 0 & -1 & 0 & \cdots & 0 & 0 & 0\\ 0 & 2 & -1 & 0 & \cdots & 0 & 0 & 0\\ -1 & -1 & 2 & -1 & \cdots & 0 & 0 & 0\\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0\\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1\\ 0 & 0 & 0 & 0 & \cdots & 0 & -2 & 2\end{pmatrix}. \tag{1}$$

It is customary, with the generalized Cartan matrix, to use the index set $(0,1,\ldots,\ell)$. The last ℓ rows and columns of **A** are the same as the Cartan matrix of the simple complex Lie algebra B_ℓ. In this subsection, we shall recall some essential properties of the Kac–Moody algebra $\mathrm{B}_\ell^{(1)}$. $\mathrm{B}_\ell^{(1)}$has ℓ+1 simple roots, which are denoted by $\alpha_0,\ldots,\alpha_\ell$. The imaginary root δ is then given by

$$\delta=\alpha_0+\alpha_1+2(\alpha_2+\cdots+\alpha_\ell) \tag{2}$$

which implies that

$$c=h\alpha_0+h\alpha_1+2(h\alpha_2+\cdots+h\alpha_\ell)\,. \tag{3}$$

For the simple roots of the Kac–Moody algebra the quantities $\langle\alpha_j,\alpha_k\rangle$ are given by

$$\langle\alpha_j,\alpha_k\rangle=\mathbf{B}_{jk} \tag{4}$$

where the matrix **B** is given below:

$$\mathbf{B}=\frac{1}{2(2\ell-1)}\begin{pmatrix}2 & 0 & -2 & 0 & \cdots & 0 & 0 & 0\\ 0 & 2 & -1 & 0 & \cdots & 0 & 0 & 0\\ -1 & -1 & 2 & -1 & \cdots & 0 & 0 & 0\\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0\\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1\\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1\end{pmatrix}. \tag{5}$$

1.3. The Lie algebra B_ℓ

B_ℓ is not only the complexification of $so(2\ell+1)$, it is also the complexification of $so(p,q)$ where $p+q=2\ell+1$. In particular, consider the algebra $so(\ell+1,\ell)$. This may be defined to be the set of real traceless $(2\ell+1)$ by $(2\ell+1)$ matrices $\mathbf{a}$ such that

$$\tilde{\mathbf{a}}\mathbf{g}+\mathbf{g}\mathbf{a}=\mathbf{0} \tag{6}$$

where $\mathbf{g}$ is a diagonal matrix with $\ell+1$ entries 1 and ℓ entries -1. With this in mind, we use a realization of B_ℓ given by

$$\Gamma(h_{\alpha_j^\circ})=\{2(\ell-1)\}^{-1}(\mathbf{X}(j)-\mathbf{X}(j+1)) \qquad \text{for } 1\leqslant j\leqslant \ell-1 \tag{7}$$

$$\Gamma(h_{\alpha_\ell^\circ})=\{2(2\ell-1)\}^{-1}\mathbf{X}(\ell) \tag{8}$$

$$\Gamma(e_{\alpha_j^\circ})=-\{2(2\ell-1)\}^{-1/2}(\mathbf{e}_{j+1,j}+\mathbf{e}_{2\ell+2-j,2\ell+1-j}) \qquad \text{for } j=1,\ldots,\ell\,. \tag{9}$$

With the above representation, (6) holds with

$$\mathbf{g}=\text{offdiag}\{1,-1,\ldots,-1,1\}\,. \tag{10}$$

(that is, $\mathbf{g}$ is an off-diagonal matrix whose entries are alternately 1 and -1, with the first and last being 1). For example if $\ell=3$ then

$$\mathbf{g}=\text{offdiag}\{1,-1,1,1,-1,1\} \tag{11}$$

thus the two 'middle' entries on the minor diagonal are the same.

The representation Γ that we have chosen is irreducible, finite-dimensional and faithful, thus fulfilling the requirements of the matrix formulation. In addition, we have that

$$\widetilde{\Gamma}=-\mathbf{g}\Gamma\mathbf{g}^{-1} \tag{12}$$

which means that the chosen representation is equivalent to its contragredient representation. This makes analysis much simpler, since the type 1b automorphisms coincide with the type 1a automorphisms, and the type 2b automorphisms coincide with the type 2a automorphisms. The Dynkin index of the above is given by

$$\gamma=\{2\ell-1\}^{-1}\,. \tag{13}$$

1.4. Notes on the matrix formulation

Consider an automorphism specified by $\{\mathbf{U}(t),u,\xi\}$. For any member of the Kac–Moody algebra it follows that

$$\tilde{\mathbf{a}}(ut)\mathbf{g}+\mathbf{g}\mathbf{a}(ut)=\mathbf{0} \tag{14}$$

and

$$\tilde{\mathbf{a}}(ut^{-1})\mathbf{g}+\mathbf{g}\mathbf{a}(ut^{-1})=\mathbf{0} \tag{15}$$

where $\mathbf{a}(t)$ is the 'matrix part' of the element, as defined in the general theory. For the subsequent analysis to work, we need a way of finding out whether

$$\tilde{\mathbf{a}}'\mathbf{g}+\mathbf{g}\mathbf{a}'=\mathbf{0} \tag{16}$$

where $\mathbf{a}'$ is defined by the following:

$$\begin{aligned} \mathbf{a}' &= \mathbf{U}(t)\mathbf{a}(ut)\mathbf{U}(t)^{-1} && \text{for type 1a automorphisms} \\ \mathbf{a}' &= \mathbf{U}(t)\mathbf{a}(ut^{-1})\mathbf{U}(t)^{-1} && \text{for type 2a automorphisms}. \end{aligned} \tag{17}$$

This requirement is a consequence of the effect of type 1a and type 2a automorphisms upon the 'matrix part' of an arbitrary algebra element. It is easily shown that condition (16) holds if and only if

$$\widetilde{\mathbf{U}(t)}\mathbf{g}\mathbf{U}(t) = f(t)\mathbf{g} \tag{18}$$

where $f(t)$ is some function of t. Given that $\mathbf{U}(t)$ is assumed to be a Laurent matrix, we may assume that

$$\det\mathbf{U}(t) = \alpha t^{\beta} \tag{19}$$

α being some non-zero complex number, and β being some integer. Then, if we take determinants of both sides of (17) above, we obtain the conclusion that $f(t) = at^{b}$, where a is some non-zero complex number, and b is some even integer.

In the subsequent sections we are going to investigate the involutive automorphisms. These sections will be as follows: section 2 will contain a study of the Weyl group of B_ℓ, which is the group of rotations of roots of B_ℓ. Section 3 contains a list of involutive automorphisms corresponding to each of the root-transformations given in section 2. (Every involutive automorphism of $B_\ell^{(1)}$ is conjugate to at least one of the automorphisms in this list.) Sections 4–7 give more detailed investigations of the conjugacy classes of the involutive automorphisms, whilst section 7 contains a summary of the results, including one representative for each conjugacy class identified in the analysis.

2. The Weyl group of B_ℓ

It was shown in the earlier papers that the only structural knowledge one needs is some familiarity with the group $\mathcal{R}$ of the corresponding simple Lie algebra, where $\mathcal{R}$ is the group of 'rotations' or root-preserving transformations of the simple Lie algebra, which in this case is B_ℓ. This stems from the fact that every conjugacy class of involutive automorphisms of the Kac–Moody algebra contains at least one Cartan-preserving automorphism, which induces, in turn, an involutive rotation τ° of B_ℓ.

For the algebra B_ℓ, the group $\mathcal{R}$ coincides with $\mathcal{W}$, the Weyl group of B_ℓ. Thus, we require one representative for each conjugacy class of involutions in $\mathcal{W}$. Since all Weyl groups are also Coxeter groups, we can make use of an algorithm developed by Richardson [5] to obtain such representatives. In this section, we shall give a list of rotations obtained in this way. The algorithm, which is easy to apply, works by finding subsets of S, the index set $\{1, 2, \ldots, \ell\}$, that satisfy the '(-1)-condition', and then determining the $\mathcal{W}$-equivalence classes of such subsets of the index set.

The number of conjugacy classes of involutions in $\mathcal{W}$ does, of course, vary with ℓ. Fortunately, there are a number of general patterns that allow us to give an exhaustive and succinct list of class representatives. For any value of ℓ, the representatives fall naturally into eight 'families', each of which contains one or more members. For those families that contain more than one member, the members all have the same overall form but incorporate one or more integral parameters. In each such case, it is to be understood that the parameters take all permitted values, thus obtaining a full set of conjugacy class representatives. We shall illustrate this as we list the representatives. The numbers 1–8 in the list below refer to the eight 'families'.

(1) This consists only of one member (the identity root-transformation τ°) where

$$\tau^\circ(\alpha_j^\circ) = \alpha_j^\circ \qquad \text{for } 1 \leqslant j \leqslant \ell. \tag{20}$$

(2) Here again, we have just one rotation and in this case it is specified by

$$\tau^\circ(\alpha_j^\circ) = -\alpha_j^\circ \qquad \text{for } 1 \leqslant j \leqslant \ell \tag{21}$$

(3) In this case the general form of τ° is given by

$$\begin{aligned} &\tau^\circ(\alpha_j^\circ) = \alpha_j^\circ && \text{for } 1 \leqslant j \leqslant q-2 \\ &\tau^\circ(\alpha_{q-1}^\circ) = \alpha_{q-1}^\circ + 2(\alpha_q^\circ + \cdots + \alpha_\ell^\circ) && \\ &\tau^\circ(\alpha_k^\circ) = -\alpha_k^\circ && \text{for } q \leqslant k \leqslant \ell\,. \end{aligned} \tag{22}$$

In (22), the integer q is assumed to take values $2 \leqslant q \leqslant \ell$ inclusive. In the special case that $q = 2$, (22) reduces to

$$\begin{aligned} &\tau^\circ(\alpha_1^\circ) = \alpha_1^\circ + 2(\alpha_2^\circ + \cdots + \alpha_\ell^\circ) \\ &\tau^\circ(\alpha_j^\circ) = -\alpha_j^\circ \qquad \text{for } 2 \leqslant j \leqslant \ell\,. \end{aligned} \tag{23}$$

(4) In this case the most general τ° is given by

$$\begin{aligned} &\tau^\circ(\alpha_j^\circ) = -\alpha_j^\circ && \text{for } j \text{ odd; } 1 \leqslant j \leqslant q \\ &\tau^\circ(\alpha_k^\circ) = \alpha_{k-1}^\circ + \alpha_k^\circ + \alpha_{k+1}^\circ && \text{for } k \text{ even; } 2 \leqslant k \leqslant q-1 \\ &\tau^\circ(\alpha_{q+1}^\circ) = \alpha_q^\circ + \alpha_{q+1}^\circ && \\ &\tau^\circ(\alpha_m^\circ) = \alpha_m^\circ && \text{for } q+2 \leqslant m \leqslant \ell\,. \end{aligned} \tag{24}$$

In (24), the parameter q is allowed to take all odd values such that $1 \leqslant 1 \leqslant \ell$. Also, for the special case $q = 1$, (24) simplifies to

$$\begin{aligned} &\tau^\circ(\alpha_1^\circ) = -\alpha_1^\circ \\ &\tau^\circ(\alpha_2^\circ) = \alpha_1^\circ + \alpha_2^\circ \\ &\tau^\circ(\alpha_m^\circ) = \alpha_m^\circ \qquad \text{for } 2 < m \leqslant \ell\,. \end{aligned} \tag{25}$$

(5) In this case, we have one parameter q, and τ° is such that

$$\begin{aligned} &\tau^\circ(\alpha_j^\circ) = \alpha_j^\circ && \text{for } 1 \leqslant j \leqslant q-2 \\ &\tau^\circ(\alpha_{q-1}^\circ) = \alpha_{q-1}^\circ + \alpha_q^\circ && \\ &\tau^\circ(\alpha_k^\circ) = -\alpha_k^\circ && \text{for } q \leqslant k \leqslant \ell;\ (\ell-k) \text{ is even} \\ &\tau^\circ(\alpha_m^\circ) = \alpha_{m-1}^\circ + \alpha_m^\circ + \alpha_{m+1}^\circ && \text{for } q < m < \ell-1;\ (\ell-m) \text{ is odd} \\ &\tau^\circ(\alpha_{\ell-1}^\circ) = \alpha_{\ell-2}^\circ + \alpha_{\ell-1}^\circ + 2\alpha_\ell^\circ\,. && \end{aligned} \tag{26}$$

In (26), q is restricted to integer values such that $1 < q < \ell$, and also such that $(\ell - q)$ is even.

(6) In this case, τ° is specified by

$$\begin{aligned} &\tau^\circ(\alpha_j^\circ) = -\alpha_j^\circ && \text{for } 1 \leqslant j \leqslant \ell;\ j \text{ odd} \\ &\tau^\circ(\alpha_k^\circ) = \alpha_{k-1}^\circ + \alpha_k^\circ + \alpha_{k+1}^\circ && \text{for } 1 < k < \ell - 2 \\ &\tau^\circ(\alpha_{\ell-1}^\circ) = \alpha_{\ell-2}^\circ + \alpha_{\ell-1}^\circ + 2\alpha_\ell^\circ . \end{aligned} \tag{27}$$

(7) Here the most general transformation is such that

$$\begin{aligned} &\tau^\circ(\alpha_j^\circ) = -\alpha_j^\circ && \text{for } 1 \leqslant j \leqslant q-2;\ j \text{ is odd} \\ &\tau^\circ(\alpha_k^\circ) = \alpha_{k-1}^\circ + \alpha_k^\circ + \alpha_{k+1}^\circ && \text{for } 2 \leqslant k \leqslant q+3;\ k \text{ is even} \\ &\tau^\circ(\alpha_{q-1}^\circ) = \alpha_{q-2}^\circ + \alpha_{q-1}^\circ + 2(\alpha_q^\circ + \cdots + \alpha_\ell^\circ) && \\ &\tau^\circ(\alpha_m^\circ) = -\alpha_m^\circ && \text{for } q \leqslant m \leqslant \ell \end{aligned} \tag{28}$$

where $1 \leqslant q \leqslant \ell - 1$ and q is odd.

(8) The final 'family' of transformations, depends upon two integer parameters q and r, with τ° given by

$$\begin{aligned} &\tau^\circ(\alpha_j^\circ) = -\alpha_j^\circ && \text{for } 1 \leqslant j \leqslant q;\ j \text{ is odd} \\ &\tau^\circ(\alpha_k^\circ) = \alpha_{k-1}^\circ + \alpha_k^\circ + \alpha_{k+1}^\circ && \text{for } 1 < k < q;\ k \text{ is even} \\ &\tau^\circ(\alpha_{q+1}^\circ) = \alpha_q^\circ + \alpha_{q+1}^\circ && \\ &\tau^\circ(\alpha_m^\circ) = \alpha_m^\circ && \text{for } q+2 \leqslant m \leqslant r-2 \\ &\tau^\circ(\alpha_{r-1}^\circ) = \alpha_{r-1}^\circ + 2(\alpha_r^\circ + \cdots + \alpha_\ell^\circ) && \\ &\tau^\circ(\alpha_n^\circ) = -\alpha_n^\circ && \text{for } r \leqslant n \leqslant \ell . \end{aligned} \tag{29}$$

3. Listing of involutive automorphisms

3.1. Plan of the section

In this section we will give a list of involutive automorphisms of $B_\ell^{(1)}$ to which all other involutive automorphisms of $B_\ell^{(1)}$ are conjugate. This list will come in three parts over the next three subsections. In particular, subsection 2 will give a list of type 1a involutive automorphisms with $u = 1$ to which all other such automorphisms are conjugate. Subsection 3 will do the same for the type 1a involutive automorphisms with $u = -1$ and subsection 4 will do this for the type 2a involutive automorphisms with $u = 1$.

In general, the listing will consist of various matrices for each of the families of root transformations given in section 2. Most of these will contain quantities such as λ_j and μ_j. Unless stated otherwise, these should be interpreted as being arbitrary with λ_j taking some non-zero complex value, and μ_j taking some integer value. Thus, for each matrix so defined, we specify a set (possibly infinite) of automorphisms of the form

$$\phi = \{\mathbf{U}(t), u, \xi\} \tag{30}$$

where the complex numbers ξ are essentially arbitrary, and we shall use the term ξ for *all* automorphisms, although these do not necessarily refer to the same value. We recall that the automorphism ϕ, as given in (30), is the same automorphism as ϕ', where $\phi' = \{\lambda t^\mu \mathbf{U}(t), u, \xi\}$. This means that we may scale each matrix $\mathbf{U}(t)$ by an arbitrary factor λt^μ. In practice, we do this such that the $(\ell+1)$th diagonal element of $\mathbf{U}(t)$ is 1. This makes subsequent analysis much simpler. In order to display the listings in a simple and compact form we shall define some submatrices. Firstly, define $\mathbf{L}_j$ and $\mathbf{L}'_j$ (where j is an integer with $1 \leqslant j \leqslant \ell$) by

$$\mathbf{L}_j = \begin{pmatrix} 0 & \lambda_j t^{\mu_j} \\ \lambda_{j+1} t^{\mu_{j+1}} & 0 \end{pmatrix} \tag{31}$$

$$\mathbf{L}'_j = \begin{pmatrix} 0 & -\lambda_j t^{\mu_j} \\ \lambda_{j+1} t^{\mu_{j+1}} & 0 \end{pmatrix}. \tag{32}$$

$\mathbf{M}_j$ and $\mathbf{M}'_j$ are defined by

$$\mathbf{M}_j = \begin{pmatrix} 0 & \lambda_j t^{\mu_j} \\ (-1)^{\mu_j} \lambda_j^{-1} t^{-\mu_j} & 0 \end{pmatrix} \tag{33}$$

$$\mathbf{M}'_j = -\begin{pmatrix} 0 & (-1)^{\mu_j} \lambda_j t^{\mu_j} \\ \lambda_j^{-1} t^{-\mu_j} & 0 \end{pmatrix}. \tag{34}$$

Similarly $\mathbf{N}_j$ and $\mathbf{N}'_j$ are given by

$$\mathbf{N}_j = \begin{pmatrix} 0 & \lambda_j t^{\mu_j} \\ \lambda_j^{-1} t^{\mu_j} & 0 \end{pmatrix} \tag{35}$$

$$\mathbf{N}'_j = \begin{pmatrix} 0 & -\lambda_j t^{-\mu_j} \\ -\lambda_j^{-1} t^{-\mu_j} & 0 \end{pmatrix}. \tag{36}$$

Furthermore, we define $\mathbf{C}_{j,k}$ and $\mathbf{C}'_{j,k}$ by

$$\mathbf{C}_{j,k} = \mathrm{diag}\{\lambda_j t^{\mu_j}, \lambda_{j+1} t^{\mu_{j+1}}, \ldots, \lambda_k t^{\mu_k}\} \tag{37}$$

$$\mathbf{C}'_{j,k} = \mathrm{diag}\{\lambda_k^{-1} t^{-\mu_k}, \lambda_{k-1}^{-1} t^{-\mu_{k-1}}, \ldots, \lambda_j^{-1} t^{-\mu_j}\} \tag{38}$$

where, in (37) and (38), we have that

$$\lambda_m^2 = 1 \qquad \text{for } j \leqslant m \leqslant k. \tag{39}$$

We define $\mathbf{C}^\circ_{j,k}$ and $\mathbf{C}'^\circ_{j,k}$ to be of the form given for $\mathbf{C}_{j,k}$ and $\mathbf{C}'_{j,k}$, but where

$$\mu_m = 0 \qquad \text{for } j \leqslant m \leqslant k \tag{40}$$

which we will describe as the t-independent forms of $\mathbf{C}_{j,k}$ and $\mathbf{C}'_{j,k}$ respectively. $\mathbf{E}_j$, $\mathbf{E}'_j$, $\mathbf{E}^\circ_j$ and $\mathbf{E}'^\circ_j$ may be defined as $\mathbf{C}_{1,j}$, $\mathbf{C}'_{1,j}$, $\mathbf{C}^\circ_{1,j}$ and $\mathbf{C}'^\circ_{1,j}$ respectively.

Similarly, we define $\mathbf{D}_j$ and $\mathbf{D}^\circ_j$, with $\mathbf{D}_j$ being described by

$$\mathbf{D}_j = \mathrm{diag}\{\lambda_j t^{\mu_j}, \ldots, \lambda_\ell t^{\mu_\ell}, 1, \lambda_\ell^{-1} t^{-\mu_\ell}, \ldots, \lambda_j^{-1} t^{-\mu_j}\} \tag{41}$$

with $\mathbf{D}^\circ_j$ being the 't-independent form' of $\mathbf{D}_j$ in the sense explained above for $\mathbf{C}^\circ_{j,k}$.

Finally, we define $\mathbf{F}_j$, $\mathbf{F}^e_j$ and $\mathbf{F}^\circ_j$. The basic form of $\mathbf{F}_j$ is given by

$$\mathbf{F}_j = \mathrm{offdiag}\{\lambda_j t^{\mu_j}, \ldots, \lambda_\ell t^{\mu_\ell}, 1, \lambda_\ell^{-1} t^{-\mu_\ell}, \ldots, \lambda_j^{-1} t^{-\mu_j}\} \tag{42}$$

with $\mathbf{F}^\circ_j$ being the t-independent form of this. $\mathbf{F}^e_j$ is as given in (42) but is such that all powers of t are even, as opposed to the t-independent form where they are all zero.

The usage of the submatrices that we have just defined will now be indicated. If $k-j$ is a positive even integer then by an expression such as '$\mathbf{L}_j, \ldots, \mathbf{L}_k$', we mean the expression $\mathrm{dsum}\{\mathbf{L}_j, \mathbf{L}_{j+2}, \ldots, \mathbf{L}_{k-2}, \mathbf{L}_k\}$ and *not* the expression $\mathrm{dsum}\{\mathbf{L}_j, \mathbf{L}_{j+1}, \ldots, \mathbf{L}_{k-1}, \mathbf{L}_k\}$. (This merely reflects the fact that $\mathbf{L}_j$ is a 2×2 square submatrix.) We make the same interpretation for all such expressions involving the 2×2 submatrices.

3.2. Involutive automorphisms of type 1a with u = 1

In this subsection we list the $\mathbf{U}(t)$, such that the corresponding type 1a automorphisms with $u = 1$ form a set of involutive automorphisms to which all other type 1a involutive automorphisms with $u = 1$ must be conjugate. We do this by working through each of the eight 'families' of root transformations that were listed in the previous section and giving the most general form of $\mathbf{U}(t)$ for each of them, thus ensuring that no automorphisms are overlooked. (The numbers 1–8 below refer to the 'families' listed in section 2.)

(1) $\mathbf{U}(t) = \mathbf{D}_1^\circ$ (43)

(2) $\mathbf{U}(t) = \mathbf{F}_1$ (44)

(3) $\mathbf{U}(t) = \mathrm{dsum}\{\mathbf{E}_{q-1}^\circ, \mathbf{F}_q, \mathbf{E}'^\circ_{q-1}\}$ (45)

(4) $\mathbf{U}(t) = \mathrm{dsum}\{\mathbf{L}_1, \ldots, \mathbf{L}_q, \mathbf{D}_{q+2}^\circ, \mathbf{L}'_q, \ldots, \mathbf{L}'_1\}$ (46)

(5) $\mathbf{U}(t) = \mathrm{dsum}\{\mathbf{E}_{q-1}^\circ, \mathbf{L}_q, \ldots, \mathbf{L}_{\ell-2}, \mathbf{F}_\ell, \mathbf{L}'_{\ell-2}, \ldots, \mathbf{L}'_q, \mathbf{E}_{q-1}^{\circ}{}'\}$ (47)

(6) $\mathbf{U}(t) = \mathrm{dsum}\{\mathbf{L}_1, \ldots, \mathbf{L}_{\ell-2}, \mathbf{F}_\ell, \mathbf{L}'_{\ell-2}, \ldots, \mathbf{L}'_1\}$ (48)

(7) $\mathbf{U}(t) = \mathrm{dsum}\{\mathbf{L}_1, \ldots, \mathbf{L}_{q-2}, \mathbf{F}_q, \mathbf{L}'_{q-2}, \ldots, \mathbf{L}'_1\}$ (49)

(8) $\mathbf{U}(t) = \mathrm{dsum}\{\mathbf{L}_1, \ldots, \mathbf{L}_q, \mathbf{C}_{q+2,r-1}^\circ, \mathbf{F}_r, \mathbf{C}_{q+2,r-1}^{\circ}{}', \mathbf{C}'^\circ_{q+2,r-1}, \mathbf{L}'_q, \ldots, \mathbf{L}'_1\}$. (50)

3.3. Involutive automorphisms of type 1a with u = − 1

This subsection will follow on in much the same fashion as the previous one.

(1) $\mathbf{U}(t) = \mathbf{D}_1^\circ$ (51)

(2) $\mathbf{U}(t) = \mathbf{F}_1^e$ (52)

(3) $\mathbf{U}(t) = \mathrm{dsum}\{\mathbf{E}_{q-1}^\circ, \mathbf{F}_q^e, \mathbf{E}'^\circ_{q-1}\}$ (53)

(4) $\mathbf{U}(t) = \mathrm{dsum}\{\mathbf{M}_1, \ldots, \mathbf{M}_q, \mathbf{D}_{q+2}^\circ, \mathbf{M}'_q, \ldots, \mathbf{M}'_1\}$ (54)

(5) $\mathbf{U}(t) = \mathrm{dsum}\{\mathbf{E}_{q-1}^\circ, \mathbf{M}_q, \ldots, \mathbf{M}_{\ell-2}, \mathbf{F}_\ell^e, \mathbf{M}'_{\ell-2}, \ldots, \mathbf{M}'_q, \mathbf{E}'^\circ_{q-1}\}$ (55)

(6) $\mathbf{U}(t) = \mathrm{dsum}\{\mathbf{M}_1, \ldots, \mathbf{M}_{\ell-2}, \mathbf{F}_\ell^e, \mathbf{M}'_{\ell-2}, \ldots, \mathbf{M}'_1\}$ (56)

(7) $\mathbf{U}(t) = \mathrm{dsum}\{\mathbf{M}_1, \ldots, \mathbf{M}_{q-2}, \mathbf{F}_q^e, \mathbf{M}'_{q-2}, \ldots, \mathbf{M}'_1\}$ (57)

(8) $\mathbf{U}(t) = \mathrm{dsum}\{\mathbf{M}_1, \ldots, \mathbf{M}_q, \mathbf{C}_{q-2,r-1}^\circ, \mathbf{F}_r^e, \mathbf{C}'^\circ_{q-2,r-1}, \mathbf{M}'_q, \ldots, \mathbf{M}'_1\}$. (58)

3.4. Involutive Automorphisms of type 2a with u = 1

We proceed in the same manner as the rest of this section, namely by working through each of the eight families of root transformations.

(1) $\mathbf{U}(t) = \mathbf{D}_1$ (59)

(2) $\mathbf{U}(t) = \mathbf{F}_1^\circ$ (60)

(3) $\mathbf{U}(t) = \mathrm{dsum}\{\mathbf{E}_{q-1}, \mathbf{F}_q^\circ, \mathbf{E}'_{q-1}\}$ (61)

(4) $\mathbf{U}(t) = \mathrm{dsum}\{\mathbf{N}_1, \ldots, \mathbf{N}_q, \mathbf{D}_{q+2}, \mathbf{N}'_q, \ldots, \mathbf{N}'_1\}$ (62)

(5) $\mathbf{U}(t) = \mathrm{dsum}\{\mathbf{E}_{q-1}, \mathbf{N}_q, \ldots, \mathbf{N}_{\ell-2}, \mathbf{F}_\ell^\circ, \mathbf{N}'_{\ell-2}, \ldots, \mathbf{N}'_q, \mathbf{E}'_{q-1}\}$ (63)

(6) $\mathbf{U}(t) = \mathrm{dsum}\{\mathbf{N}_1, \ldots, \mathbf{N}_{\ell-2}, \mathbf{F}_\ell^\circ, \mathbf{N}'_{\ell-2}, \ldots, \mathbf{N}'_1\}$ (64)

(7) $\mathbf{U}(t) = \mathrm{dsum}\{\mathbf{N}_1, \ldots, \mathbf{N}_{q-2}, \mathbf{F}_q^\circ, \mathbf{N}'_{q-2}, \ldots, \mathbf{N}'_1\}$ (65)

(8) $\mathbf{U}(t) = \mathrm{dsum}\{\mathbf{N}_1, \ldots, \mathbf{N}_q, \mathbf{C}_{q+2,r-1}, \mathbf{F}_r^\circ, \mathbf{C}'_{q+2,r-1}, \mathbf{N}'_q, \ldots, \mathbf{N}'_1\}$ (66)

4. Simplification of the forms of $\mathbf{U}(t)$

In the previous section we listed a number of matrices $\mathbf{U}(t)$, with the intention of studying the automorphisms corresponding to them and, in particular, of determining the conjugacy classes of the involutive automorphisms within the group of all automorphisms of the algebra. In this section we will show that there is a subset of the automorphisms given in section 3, such that all of the other involutive automorphisms are conjugate to the members of this subset.

(i) Consider a matrix $\mathbf{U}(t)$ which is of the form

$$\mathbf{U}(t) = \mathrm{dsum}\{\mathbf{H}, \mathbf{L}_j, \mathbf{H}', \mathbf{L}_j{}', \mathbf{H}''\} \tag{67}$$

where $\mathbf{H}$, $\mathbf{H}'$ and $\mathbf{H}''$ are arbitrary square matrices, and $\mathbf{H}$ and $\mathbf{H}''$ are of the same dimension, namely $j-1$. Then, define $\mathbf{U}'(t)$ to be the matrix such that

$$\mathbf{U}'(t) = \mathrm{dsum}\{\mathbf{H}, \mathbf{w}, \mathbf{H}', -\mathbf{w}, \mathbf{H}''\} \tag{68}$$

where $\mathbf{w} = \mathrm{diag}\{1, -1\}$ is 2×2. We shall now demonstrate that all type 1a automorphisms $\{\mathbf{U}(t), u, \xi\}$, where $\mathbf{U}(t)$ is as given in equation (67) are conjugate to the type 1a involutive automorphism $\{\mathbf{U}'(t), u, \xi\}$. With this in mind, define $\mathbf{v}(t)$ by

$$\mathbf{v}(t) = \mathrm{dsum}\{\mathbf{1}_{j-1}, \mathbf{t}_1, \mathbf{1}_{2\ell-2j+1}, \mathbf{t}_2, \mathbf{1}_{j-1}\} \tag{69}$$

$$\text{where } \mathbf{t}_1 = \frac{\mathrm{i}}{\sqrt{2}}\begin{pmatrix} \lambda_j^{-1} t^{-\mu_j} & 1 \\ \lambda_j^{-1} t^{-\mu_j} & -1 \end{pmatrix} \qquad \mathbf{t}_2 = \frac{\mathrm{i}}{\sqrt{2}}\begin{pmatrix} 1 & \lambda_j t^{\mu_j} \\ 1 & -\lambda_j t^{\mu_j} \end{pmatrix}. \tag{70}$$

It is easy to show that

$$\widetilde{\mathbf{v}}(t)\mathbf{g}\mathbf{v}(t) = \mathbf{g} \tag{71}$$

$$\mathbf{v}(t)\mathbf{U}(t)\mathbf{v}(t)^{-1} = \mathbf{U}'(t) \tag{72}$$

which proves the assertion we have just made.

Similarly, suppose that $\mathbf{U}(t)$ is as given in (67) but contains $\mathbf{M}_j$ and $\mathbf{M}_j'$ instead of $\mathbf{L}_j$ and $\mathbf{L}_j'$. With $\mathbf{v}(t)$ as defined in (69) we have

$$\mathbf{v}(t)\mathbf{U}(t)\mathbf{v}(-t)^{-1} = \mathbf{U}'(t) \tag{73}$$

which means that all of the type 1a automorphisms $\{\mathbf{U}(t), -1, \xi\}$, where $\mathbf{U}(t)$ is of the form (67) (but with $\mathbf{L}_j$ replaced by $\mathbf{M}_j$), are conjugate to the type 1a automorphism $\{\mathbf{U}'(t), -1, \xi\}$. Similarly, we may replace $\mathbf{L}_j$ in (67) by $\mathbf{N}_j$ and infer that all of the type 2a automorphisms of the form $\{\mathbf{U}(t), 1, \xi\}$ are conjugate to the type 2a automorphism given by $\{\mathbf{U}'(t), 1, \xi\}$. This simplifies matters greatly.

(ii) Consider now $\mathbf{U}(t)$ given by

$$\mathbf{U}(t) = \mathrm{dsum}\{\mathbf{H}, \mathbf{C}_{j,k}, \mathbf{H}, \mathbf{C}'_{j,k}, \mathbf{H}''\} \tag{74}$$

where $\mathbf{H}$ and $\mathbf{H}''$ are arbitrary $(j-1)\times(j-1)$ matrices and $\mathbf{H}'$ is an arbitrary square matrix of dimension $2\ell - 2k + 1$. Define $\mathbf{U}'(t)$ to be

$$\mathbf{U}'(t) = \mathrm{dsum}\{\mathbf{H}, \mathbf{w}, \mathbf{H}', \mathbf{w}', \mathbf{H}'\} \tag{75}$$

where, in (75), $\mathbf{w}$ is identical to $\mathbf{C}_{j,k}$ except that two of its diagonal elements ($\lambda_m t^{\mu_m}$ and $\lambda_n t^{\mu_n}$) have been interchanged. The matrix $\mathbf{w}'$ is obtained from $\mathbf{C}'_{j,k}$ by performing the 'same' interchange of index sets so that $\lambda_m^{-1}t^{-\mu_m}$ and $\lambda_n^{-1}t^{-\mu_n}$ are to be exchanged. Then, for two automorphisms of the same type, $\{\mathbf{U}(t), u, \xi\}$ and $\{\mathbf{U}'(t), u, \xi\}$ are conjugate. This means that the order of the elements in $\mathbf{C}_{j,k}$ is essentially arbitrary, provided that we perform the appropriate permutation on the elements of $\mathbf{C}'_{j,k}$ at the same time. We will now demonstrate this.

There are two possible cases. In the first $m - n$ is even, which means that there are an odd number of elements between $\lambda_m t^{\mu_m}$ and $\lambda_n t^{\mu_n}$. Similarly for $\lambda_m^{-1}t^{-\mu_m}$ and $\lambda_n^{-1}t^{-\mu_n}$. Define a matrix $\mathbf{v}(t)$ (which is t-independent) by

$$v_{aa} = \begin{cases} 1 & \text{if } a \neq m, n, 2\ell+2-m, 2\ell+2-n \\ 0 & \text{if } a = m, n, 2\ell+2-m, 2\ell+2-n \end{cases}$$
$$v_{ab} = v_{ba} = \begin{cases} 1 & \text{if a} = m, b = n \\ 1 & \text{if } a = 2\ell+2-m, b = 2\ell+2-n \end{cases} \tag{76}$$

with all other entries being zero. Then $\tilde{\mathbf{v}}\mathbf{g}\mathbf{v} = \mathbf{g}$, and

$$\mathbf{v}\mathbf{U}(t)\mathbf{v}^{-1} = \mathbf{U}'(t) \tag{77}$$

This proves our assertion when $m - n$ is even. Now suppose that $m - n$ is odd, and define a t-independent $\mathbf{v}$ by the following:

$$v_{aa} = \begin{cases} i & \text{if } a \neq m, n, 2\ell+2-m, 2\ell+2-n \\ 0 & \text{if } a = m, n, 2\ell+2-m, 2\ell+2-n \end{cases}$$
$$v_{ab} = v_{ba} = \begin{cases} 1 & \text{if } a = m, b = n \\ 0 & \text{if } a = 2\ell+2-m, b = 2\ell+2-n \end{cases} \tag{78}$$

with all other entries being zero. Then $\tilde{\mathbf{v}}\mathbf{g}\mathbf{v} = -\mathbf{g}$, and

$$\mathbf{v}\mathbf{U}(t)\mathbf{v}^{-1} = \mathbf{U}'(t) \tag{79}$$

where $\mathbf{U}'(t)$ is as defined in (75). This proves our assertion when $m - n$ is some odd integer. Clearly, repeated application of the above argument shows that the position of such interchanged diagonal elements is arbitrary.

(iii) Now, suppose that in (74) we were to replace $\mathbf{U}(t)$ with the matrix $\mathbf{U}(t)$ given by the following:

$$\mathbf{U}(t) = \begin{pmatrix} \mathbf{H} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}'' \end{pmatrix}. \tag{80}$$

We may interchange the elements $\lambda_m t^{\mu_m}$ and $\lambda_n t^{\mu_n}$ of $\mathbf{F}_j$, and the elements $\lambda_m^{-1}t^{-\mu_m}$ and $\lambda_n^{-1}t^{-\mu_n}$ of $\mathbf{F}_j$ using precisely the same techniques as were explained above. Hence, we may alter the index set of the off-diagonal elements arbitrarily, as we may do for the diagonal elements.

(iv) Let $\mathbf{U}(t)$ be as given in (80). We shall show here that it is possible to assume without loss of generality that

$$\lambda_k = 1 \qquad \text{for } j \leqslant k \leqslant \ell. \tag{81}$$

Let $\mathbf{U}'(t)$ be obtained from $\mathbf{U}(t)$ by setting $\lambda_k = 1$, where $j \leqslant k \leqslant \ell$, and let $\mathbf{s}$ be the t-independent matrix

$$\mathbf{s} = \text{dsum}\{\mathbf{1}_{j-1}, \mathbf{x}, \mathbf{1}_{j-1}\} \tag{82}$$

where $\mathbf{x} = \mathrm{diag}\{\lambda_j{}^{1/2}, \ldots, \lambda_\ell{}^{1/2}, 1, \lambda_\ell{}^{-1/2}, \ldots, \lambda_j{}^{-1/2}\}$. Then

$$\tilde{\mathbf{s}}\mathbf{g}\mathbf{s} = \mathbf{g} \qquad \mathbf{s}\mathbf{U}(t)'\mathbf{s}^{-1} = \mathbf{U}(t) \tag{83}$$

implying that $\{\mathbf{U}(t), u, \xi\}$ is conjugate to $\{\mathbf{U}(t)', u, \xi\}$.

(v) In the previous paragraph we dealt with the parameters λ_k that occur in the submatrices $\mathbf{F}_j$. In this subsection we are going to look at the integer variables μ_k in the same submatrices. In particular, if $\{\mathbf{U}(t), \pm 1, \xi\}$ is a type 1a automorphism, then we may assume

$$\mu_k = \begin{cases} 0 & \text{if } \mu_k \text{ is even} \\ 1 & \text{if } \mu_k \text{ is odd} \end{cases} \tag{84}$$

that is, only the parity of μ_k is important, and not the absolute value thereof. (We are concerned here only with type 1a automorphisms. This is because submatrices $\mathbf{F}_j$ occuring in section 3.4 are all of the form $\mathbf{F}_j^0$, for which all of the integer parameters are already zero.) It is assumed that $\mathbf{U}(t)$ is still of the form given by (80).

We define quantities ρ_k according to the prescription

$$\rho_k = \begin{cases} 0 & \text{if } \rho_k \text{ is even} \\ 1 & \text{if } \rho_k \text{ is odd} \end{cases} \tag{85}$$

where $j \leqslant k \leqslant \ell$. Define the matrix $\mathbf{U}'(t)$ by

$$\mathbf{U}'(t) = \mathrm{dsum}\{\mathbf{H}, \mathbf{w}', \mathbf{H}''\} \qquad \mathbf{w}' = \mathrm{offdiag}\{t^{\rho_j}, \ldots, t^{\rho_\ell}, 1, t^{-\rho_\ell}, \ldots, t^{-\rho_j}\}. \tag{86}$$

Define $\mathbf{s}(t)$ by

$$\begin{aligned} \mathbf{s}(t) &= \mathrm{dsum}\{\mathbf{1}_{j-1}, \mathbf{x}(t), \mathbf{1}_{j-1}\} \\ \mathbf{x}(t) &= \mathrm{diag}\{t^{(\mu_j - \rho_j)/2}, \ldots, t^{(\mu_\ell - \rho_\ell)/2}, 1, t^{(\mu_\ell - \rho_\ell)/2}, \ldots, t^{(\mu_j - \rho_j)/2}\}. \end{aligned} \tag{87}$$

Then $\widetilde{\mathbf{s}(t)}\mathbf{g}\mathbf{s}(t) = \mathbf{g}$. If $\{\mathbf{U}'(t), 1, \xi\}$ is a type 1a automorphism, then we also have that

$$\mathbf{s}(t)\mathbf{U}'(t)\mathbf{s}(t)^{-1} = \mathbf{U}(t) \tag{88}$$

where $\mathbf{U}(t)$ is as given in (80), but with $\lambda_k = 1$ for $j \leqslant k \leqslant \ell$. The previous subsection tells us that λ_k is arbitrary anyway. Also, with $\mathbf{U}''(t)$ defined as

$$\begin{aligned} \mathbf{U}''(t) &= \mathrm{dsum}\{\mathbf{H}, \mathbf{w}'', \mathbf{H}''\} \\ \mathbf{w}'' &= \mathrm{offdiag}\{(-1)^{\mu_j/2}, \ldots, (-1)^{\mu_\ell/2}, 1, (-1)^{\mu_\ell/2}, \ldots, (-1)^{\mu_j/2}\} \end{aligned} \tag{89}$$

and $\mathbf{s}(t)$ as in (87) we have that

$$\mathbf{s}(t)\mathbf{U}''(t)\mathbf{s}(-t)^{-1} = \mathbf{U}(t) \tag{90}$$

where $\mathbf{U}(t)$ is as given by (80), but with $\lambda_k = 1$ for k such that $j \leqslant k \leqslant \ell$.

(vi) Let $\{\mathbf{U}(t), 1, \xi\}$ be a type 2a automorphism, where $\mathbf{U}(t)$ is given by

$$\mathbf{U}(t) = \mathrm{dsum}\{\mathbf{H}, \mathbf{C}_{j,k}, \mathbf{H}', \mathbf{C}'_{j,k}, \mathbf{H}''\} \tag{91}$$

where $\mathbf{H}$, $\mathbf{H}'$ are square and of dimension $j-1$, and $\mathbf{H}''$ is square and of dimension $2\ell-2k+1$. Let ρ_m be as defined in (85). Then one can assume without loss of generality that

$$\mu_m = \rho_m \qquad \text{for } j \leqslant m \leqslant k\,. \tag{92}$$

Define $\mathbf{s}(t)$ by

$$\begin{aligned} \mathbf{s}(t) &= \mathrm{dsum}\{\mathbf{x}(t), \mathbf{1}_{2\ell-2k+1}, \mathbf{x}'(t)\} \\ \mathbf{x}(t) &= \mathrm{diag}\{t^{(\mu_j-\rho_j)/2}, \ldots, t^{(\mu_k-\rho_k)/2}\} \\ \mathbf{x}'(t) &= \mathrm{diag}\{t^{(\rho_k-\mu_k)/2}, \ldots, t^{(\rho_j-\mu_j)/2}\} \end{aligned} \tag{93}$$

so that $\widetilde{\mathbf{s}(t)}\mathbf{g}\mathbf{s}(t) = \mathbf{g}$. With $\mathbf{U}'(t)$ given by (91), but with

$$\mu_m = \rho_m \qquad \text{for } j \leqslant m \leqslant k \tag{94}$$

then

$$\mathbf{s}(t)\mathbf{U}'(t)\mathbf{s}(t)^{-1} = \mathbf{U}(t) \tag{95}$$

and so the assumption in (92) is valid.

(vii) Let $\mathbf{U}(t)$ be a matrix of the form

$$\mathbf{U}(t) = \begin{pmatrix} \mathbf{H} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{w} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{w} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H}'' \end{pmatrix} \tag{96}$$

where, in the above, $\mathbf{H}$ and $\mathbf{H}'$ are arbitrary matrices of the same size, say $m \times m$, and $\mathbf{H}$ is an $n \times n$ arbitrary square matrix whose form need not concern us here. More importantly, $\mathbf{w}$ is the 2×2 matrix given by

$$\mathbf{w} = \mathrm{offdiag}\{1, 1\}\,. \tag{97}$$

Let us now define a matrix $\mathbf{U}'(t)$ by

$$\mathbf{U}'(t) = \mathrm{dsum}\{\mathbf{H}, \mathbf{x}, \mathbf{H}', -\mathbf{x}, \mathbf{H}''\} \qquad \mathbf{x} = \mathrm{diag}\{1, -1\}\,. \tag{98}$$

Then $\{\mathbf{U}(t), u, \xi\}$ is conjugate to $\{\mathbf{U}'(t), u, \xi\}$, as we now demonstrate. Define a t-independent matrix $\mathbf{s}$ by

$$\mathbf{s} = \begin{pmatrix} \mathbf{1}_m & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{t}_1 & \mathbf{0} & \mathbf{t}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{t}_3 & \mathbf{0} & \mathbf{t}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_m \end{pmatrix} \tag{99}$$

where

$$\mathbf{t}_1 = \tfrac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \qquad \mathbf{t}_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \mathbf{t}_3 = -\tfrac{1}{4}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \mathbf{t}_4 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{100}$$

Thus, $\tilde{\mathbf{s}}\mathbf{g}\mathbf{s} = \mathbf{g}$, and furthermore

$$\mathbf{s}\mathbf{U}'(t)\mathbf{s}^{-1} = \mathbf{U}(t) \tag{101}$$

which is all we require.

(viii) Let $\mathbf{U}(t)$ be of the form

$$\mathbf{U}(t) = \begin{pmatrix} \mathbf{v}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{w}_1 & \mathbf{0} & \mathbf{w}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{w}_3 & \mathbf{0} & \mathbf{w}_4 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{v}_2 \end{pmatrix} \tag{102}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are arbitrary square matrices (of order $m \times m$ say), and $\mathbf{w}_p$ for $p = 1, \ldots, 4$ are arbitrary square matrices (of order $n \times n$ say). With $\mathbf{U}'(t)$ defined by

$$\mathbf{U}'(t) = \begin{pmatrix} \mathbf{v}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{w}_1 & \mathbf{0} & \mathbf{w}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{1}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{w}_3 & \mathbf{0} & \mathbf{w}_4 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{v}_2 \end{pmatrix} \tag{103}$$

then it may be shown that there exists a matrix $\mathbf{s}$, which is t-independent and is such that

$$\tilde{\mathbf{s}}\mathbf{g}\mathbf{s} = \mathbf{g} \qquad \mathbf{s}\mathbf{U}'(t)\mathbf{s}^{-1} = \mathbf{U}(t)\,. \tag{104}$$

If n is even then

$$\mathbf{s} = \begin{pmatrix} \mathbf{1}_m & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathrm{i}/\sqrt{2} & \mathbf{0} & \mathrm{i}/\sqrt{2} & \mathbf{0} & \mathrm{i}/\sqrt{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_n & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -1/\sqrt{2} & \mathbf{0} & 0 & \mathbf{0} & 1/\sqrt{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathrm{i}/\sqrt{2} & \mathbf{0} & \mathrm{i}/\sqrt{2} & \mathbf{0} & -\mathrm{i}/\sqrt{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_m \end{pmatrix} \tag{105}$$

and if n is odd we let

$$\mathbf{s} = \begin{pmatrix} \mathbf{1}_m & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1/2 & \mathbf{0} & \mathrm{i}/\sqrt{2} & \mathbf{0} & 1/2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_n & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathrm{i}/\sqrt{2} & \mathbf{0} & 0 & \mathbf{0} & -\mathrm{i}/\sqrt{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1/2 & \mathbf{0} & -\mathrm{i}/\sqrt{2} & \mathbf{0} & 1/2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_m \end{pmatrix} \tag{106}$$

which satisfy (104) as required.

(ix) Let s be some arbitrary non-zero complex number, and consider the mapping

$$\mathbf{U}(t) \mapsto \mathbf{U}(st)\,. \tag{107}$$

This mapping has the same effect as altering the coefficients λ_j in $\mathbf{U}(t)$, which we have seen to be arbitrary themselves. Thus, if $\{\mathbf{U}(t), 1, \xi\}$ and $\{\mathbf{U}'(t), 1, \xi\}$ are type 1a automorphisms then they are conjugate. We may assume then that $s = 1$.

(x) The analysis here is similar to that in (vii), and concerns those type 1a automorphisms $\{\mathbf{U}(t), 1, \xi\}$ where $\mathbf{U}(t)$ is of the form

$$\mathbf{U}(t) = \begin{pmatrix} \mathbf{v}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{x}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{v}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{x}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{v}_3 \end{pmatrix} \tag{108}$$

where $\mathbf{x}_1 = \text{offdiag}\{t, t\}$, $\mathbf{x}_2 = \text{diag}\{t^{-1}, t^{-1}\}$, and $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are arbitrary square matrices, with dimensions m, n and m respectively. Note that m is less that $\ell - 1$. Define $\mathbf{U}'(t)$ by the following:

$$\mathbf{U}'(t) = \text{dsum}\{\mathbf{v}_1, \mathbf{w}, \mathbf{v}_2, -\mathbf{w}, \mathbf{v}_3\} \tag{109}$$

where $\mathbf{w}$=diag$\{1, -1\}$ is 2×2. Then there exists a matrix $\mathbf{s}(t)$ such that

$$\widetilde{\mathbf{s}(t)}\mathbf{g}\mathbf{s}(t) = \mathbf{g} \qquad \mathbf{s}(t)\mathbf{U}'(t)\mathbf{s}^{-1} = \mathbf{U}(t) \tag{110}$$

a suitable choice of $\mathbf{s}(t)$ being

$$\mathbf{s}(t) = \begin{pmatrix} \mathbf{1}_m & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{t}_1 & \mathbf{0} & \mathbf{t}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{t}_3 & \mathbf{0} & \mathbf{t}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_m \end{pmatrix} . \tag{111}$$

5. Study of the conjugacy classes of the type 1a involutive automorphisms with $u = 1$

This section deals with those involutive automorphisms $\{\mathbf{U}(t), 1, \xi_u\}$, where $\mathbf{U}(t)$ is given by one of (43)–(50). The previous section explained how various assumptions could be made about $\mathbf{U}(t)$, and we shall make use of that analysis here.

It is useful to recall briefly the necessary and sufficient conditions for the two involutive automorphisms $\{\mathbf{U}_1(t), 1, \xi_1\}$ and $\{\mathbf{U}_2, 1, \xi_2\}$ to be conjugate. If this is the case, then one of the following must hold:

$$\lambda t^{\mu}\mathbf{U}_1(t) = \mathbf{S}(t)\mathbf{U}_2(st)\mathbf{S}(t)^{-1} \qquad \lambda t^{\mu}\mathbf{U}_1(t) = \mathbf{S}(t)\mathbf{U}_2(st^{-1})\mathbf{S}(t)^{-1} . \tag{112}$$

Clearly, in the above, $\mathbf{S}(t)$ must satisfy (18).

Systematically applying the results of section 4 yields a subset of the automorphisms that is smaller than that listed in section 3.2, to which all other type 1a automorphisms with $u = 1$ are conjugate. This subset consists of the automorphisms $\{\mathbf{U}(t), 1, \xi\}$, where $\mathbf{U}(t)$ is of one of the following two forms:

$$\mathbf{U}(t) = \text{dsum}\{\mathbf{1}_{N_+}, \mathbf{1}_{N_-}, \mathbf{1}_1, -\mathbf{1}_{N_-}, -\mathbf{1}_{N_+}\} \tag{113}$$

$$\mathbf{U}(t) = \text{dsum}\{\mathbf{1}_p, -\mathbf{1}_{\ell-p-1}, \text{offdiag}\{t, 1, t^{-1}\}, -\mathbf{1}_{\ell-p-1}, \mathbf{1}_p\} \tag{114}$$

where, in (113), N_+ and N_- are non-negative integers such that $N_+ + N_- = \ell$, and in (113) p is some non-negative integer that is less than ℓ. (p is to be interpreted as a variable, being allowed to take integer values from 0 to $\ell - 1$.)

Let us start by looking at the automorphisms $\{\mathbf{U}(t), 1, 0\}$, where $\mathbf{U}(t)$ is given by (113). Clearly there are only $(\ell + 1)$ of these. Let us denote by $\mathbf{U}_j^\circ$ the matrix given by $\mathbf{U}(t)$ in (113) for which $N_- = j$. What we want to know, of course, is into how many disjoint conjugacy classes these automorphisms fall. Quite conveniently, the answer turns out to be $(\ell + 1)$. For, suppose that the following were to hold:

$$\mathbf{S}\mathbf{U}_j^\circ\mathbf{S}^{-1} = \lambda t^\mu \mathbf{U}_k^\circ . \tag{115}$$

Then it can easily be verified that there exists no non-singular matrix $\mathbf{S}$ such that (115) holds for $j \neq k$ and so this proves that each of the automorphisms under consideration is disjoint from all of the others. That is, the automorphisms $\{\mathbf{U}(t), 1, 0\}$ where $\mathbf{U}(t)$ is given by (113) fall into precisely $(\ell + 1)$ conjugacy classes. We denote the conjugacy classes by

$$(A)^{(0)}, \ldots, (A)^{(\ell)}$$

and define $(A)^{(m)}$ for $1 \leqslant m \leqslant \ell$ to contain the automorphism $\{\mathbf{U}_m^\circ, 1, 0\}$.

We are thus left with the automorphisms $\{\mathbf{U}(t), 1, \xi\}$, where $\mathbf{U}(t)$ is of the form (114). Clearly there are only ℓ such matrices, and hence ℓ such automorphisms. Let us denote by $\mathbf{U}_m(t)$ the matrix given by (114) which is such that $p = m$. It will be shown ultimately that each of these automorphisms is disjoint from the others, and also that they are disjoint from those that we have just investigated, thus giving us a total of $(2\ell + 1)$ conjugacy classes of type 1a involutive automorphisms with $u = 1$.

To begin with, we must show that $\{\mathbf{U}_m(t), 1, \xi_m\}$ is disjoint from $\{\mathbf{U}_n, 1, \xi_n\}$, where $m \neq n$. This follows from the fact that there exists no non-singular matrix $\mathbf{S}(t)$ such that

$$\mathbf{S}(t)\mathbf{U}_m(t)\mathbf{S}(t)^{-1} = \lambda t^\mu \mathbf{U}_n(t). \tag{116}$$

(An easy way to check is to try to find such a matrix for the special case $t = 1$, which proves to be impossible.)

Thus we are left with ℓ mutually disjoint automorphisms which may, or may not, belong to the conjugacy classes $(A)^{(j)}$ for $0 \leqslant j \leqslant \ell$. As it turns out, they do not belong to these classes, although proof of this is not quite straightforward. Prior to actually proving that this is the case, it is wise to give a helpful lemma, and its basic proof.

Lemma 1. Let $\mathbf{U}_1(t)$ and $\mathbf{U}_2(t)$ be two matrices that are related to each other by

$$\lambda t^\mu \mathbf{U}_1(t) = \mathbf{R}(t)\mathbf{U}_2(t)\mathbf{R}(t)^{-1} \tag{117}$$

where $\mathbf{R}(t)$ is a Laurent matrix such that

$$\widetilde{\mathbf{R}(t)}\mathbf{g}\mathbf{R}(t) \neq \alpha t^\beta \mathbf{g} . \tag{118}$$

Suppose also that there does exist some Laurent matrix $\mathbf{S}(t)$ such that

$$\lambda t^\mu \mathbf{U}_1(t) = \mathbf{S}(t)\mathbf{U}_2(t)\mathbf{S}(t)^{-1} \tag{119}$$

$$\widetilde{\mathbf{S}(t)}\mathbf{g}\mathbf{S}(t) = \alpha t^\beta \mathbf{g} \tag{120}$$

for some non-zero complex number α, and some integer β. Then

$$\mathbf{S}(t) = \mathbf{R}(t)\mathbf{Q}(t) \tag{121}$$

where $\mathbf{Q}(t)$ is also a Laurent matrix, and is such that

$$\mathbf{Q}(t)\mathbf{U}_2(t)\mathbf{Q}(t)^{-1} = \mathbf{U}_2(t) . \tag{122}$$

Proof. From (119), we have that $\mathbf{S}(t) = \lambda t^{\mu}\mathbf{U}_1(t)\mathbf{S}(t)\mathbf{U}_2(t)^{-1}$, and we may substitute (117) into this, giving $\mathbf{S}(t) = \mathbf{R}(t)\mathbf{Q}(t)$, where the matrix $\mathbf{Q}(t)$ is given by

$$\mathbf{Q}(t) = \mathbf{U}_2(t)\mathbf{R}(t)^{-1}\mathbf{S}(t)\mathbf{U}_2(t)^{-1}\,. \tag{123}$$

It follows, from the properties assumed already, that $\mathbf{Q}(t)$ is indeed a Laurent matrix, and satisfies (122).

Now recall that, if $\{\mathbf{U}_1^{\circ}, 1, 0\}$ belongs to $(A)^{(j)}$ for some j, and if $\{\mathbf{U}_2^{\circ}, 1, 0\}$ belongs to $(A)^{(k)}$ for $j \neq k$ then there does not exist any non-singular matrix $\mathbf{m}$ such that

$$\mathbf{m}\mathbf{U}_1^{\circ}\mathbf{m}^{-1} = \lambda\mathbf{U}_2^{\circ} \tag{124}$$

for any non-zero complex number λ. If $\mathbf{U}_m(t)$ and $\mathbf{U}_n(t)$ are as defined in (114) then consider $\mathbf{U}_m(1)$, which is clearly t-independent and so we may infer from section 4 that there exists some t-independent matrix $\mathbf{n}$ such that

$$\mathbf{n}\mathbf{U}_m(1)\mathbf{n}^{-1} = \lambda\mathbf{U}_{m+1}^{\circ} \tag{125}$$

where $\mathbf{U}_{m+1}^{\circ}$ is of the form (113). It is then immediately clear that there are precisely two possibilities, namely that:

(1) $\{\mathbf{U}_m(t), 1, \xi\}$ belongs to the conjugacy class $(A)^{(m+1)}$, or that
(2) $\{\mathbf{U}_m(t), 1, \xi\}$ belongs to some conjugacy class disjoint from those that we have identified already.

Using the lemma above, we shall demonstrate that the second of these is the one that holds. We assume, by way of obtaining a contradiction, that we have

$$\mathbf{S}(t)\mathbf{U}_{m+1}^{\circ}\mathbf{S}(t)^{-1} = \lambda t^{\mu}\mathbf{U}_m(t) \tag{126}$$

$$\widetilde{\mathbf{S}(t)}\mathbf{g}\mathbf{S}(t) = \alpha t^{\beta}\mathbf{g}\,. \tag{127}$$

Using the notation of the lemma, we state that there exists a matrix $\mathbf{R}(t)$ which is such that

$$\mathbf{R}(t)\mathbf{U}_{m+1}^{\circ}\mathbf{R}(t)^{-1} = -\mathbf{U}_m(t) \tag{128}$$

$$\widetilde{\mathbf{R}}(t)\mathbf{g}\mathbf{R}(t) = \mathrm{diag}\{1, \ldots, 1, t^{-1}, t^{-1}, 1, \ldots, 1\} \tag{129}$$

where the t^{-1} elements in (129) occur in the ℓth and the $(\ell + 1)$th places. In fact, such a matrix $\mathbf{R}(t)$ is given by

$$\mathbf{R}(t) = \begin{pmatrix} \mathbf{t}_1 & \mathbf{0} & \mathbf{t}_2 \\ \mathbf{0} & \mathbf{v}(t) & \mathbf{0} \\ \mathbf{t}_3 & \mathbf{0} & \mathbf{t}_4 \end{pmatrix} \tag{130}$$

where the submatrices are defined by

$$\begin{aligned}
\mathbf{t}_1 &= 2^{-1/2}\mathrm{diag}\{1, \mathrm{i}, 1, \mathrm{i}, \ldots\} && \text{i.e. alternating } 1, i \text{ from the corner} \\
\mathbf{t}_2 &= 2^{-1/2}\mathrm{offdiag}\{\mathrm{i}, -1, \mathrm{i}, -1, \ldots\} && \text{i.e. alternating i}, -1 \text{ from the corner} \\
\mathbf{t}_3 &= 2^{-1/2}\mathrm{offdiag}\{\ldots, \mathrm{i}, 1, \mathrm{i}, 1\} && \text{i.e. alternating i}, 1 \text{ from the corner} \\
\mathbf{t}_4 &= 2^{-1/2}\mathrm{diag}\{\ldots, 1, -\mathrm{i}, 1, -\mathrm{i}\} && \text{i.e. alternating } 1, -\mathrm{i} \text{ from the corner}
\end{aligned} \tag{131}$$

and $\mathbf{V}(t)$ is defined by

$$\mathbf{V}(t) = \frac{\alpha}{\sqrt{2}} \begin{pmatrix} \mathrm{i} & -1 & 0 \\ 0 & 0 & -1 \\ \mathrm{i}t^{-1} & t^{-1} & 0 \end{pmatrix} \tag{132}$$

with α taking values 1 if ℓ is even and i if ℓ is odd. Thus, the conditions of the lemma are satisfied, and we infer that

$$\mathbf{S}(t) = \mathbf{R}(t)\mathbf{Q}(t)\,. \tag{133}$$

Here $\mathbf{Q}(t)$ must be a Laurent matrix, and must also be such that

$$\mathbf{Q}(t)\mathbf{U}^\circ_{m+1}\mathbf{Q}(t)^{-1} = \eta\mathbf{U}^\circ_{m+1}\,. \tag{134}$$

However, (134) is true only if $\eta = 1$, and then $\mathbf{Q}(t)$ is of the form:

$$\mathbf{Q}(t) = \begin{pmatrix} \mathbf{H}_{11} & \mathbf{0} & \mathbf{H}_{13} & \mathbf{0} & \mathbf{H}_{15} & \mathbf{0} & \mathbf{H}_{17} \\ \mathbf{0} & \mathbf{K}_{22} & \mathbf{0} & \mathbf{K}_{24} & \mathbf{0} & \mathbf{K}_{26} & \mathbf{0} \\ \mathbf{H}_{31} & \mathbf{0} & \mathbf{H}_{33} & \mathbf{0} & \mathbf{H}_{35} & \mathbf{0} & \mathbf{H}_{37} \\ \mathbf{0} & \mathbf{K}_{42} & \mathbf{0} & \mathbf{K}_{44} & \mathbf{0} & \mathbf{K}_{46} & \mathbf{0} \\ \mathbf{H}_{51} & \mathbf{0} & \mathbf{H}_{53} & \mathbf{0} & \mathbf{H}_{55} & \mathbf{0} & \mathbf{H}_{57} \\ \mathbf{0} & \mathbf{K}_{62} & \mathbf{0} & \mathbf{K}_{64} & \mathbf{0} & \mathbf{K}_{66} & \mathbf{0} \\ \mathbf{H}_{71} & \mathbf{0} & \mathbf{H}_{73} & \mathbf{0} & \mathbf{H}_{75} & \mathbf{0} & \mathbf{H}_{77} \end{pmatrix} \tag{135}$$

where the non-zero submatrices in the above are entirely arbitrary and are square. (Here $\mathbf{Q}(t)$ is partitioned by dividing its index set thus: the first m rows/columns, the next $\ell-m-1$ rows/columns the next row/column, then the next and the next; the next $\ell-m-1$ rows/columns and finally the last m rows/columns.)

Obviously $\mathbf{Q}(t)$ is decomposable, and by a suitable re-ordering of its index set could be expressed as

$$\mathbf{Q}(t) = \begin{pmatrix} \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{pmatrix} \tag{136}$$

with $\mathbf{H}$,$\mathbf{K}$ being given by

$$\mathbf{H} = \begin{pmatrix} \mathbf{H}_{11} & \mathbf{H}_{13} & \mathbf{H}_{15} & \mathbf{H}_{17} \\ \mathbf{H}_{31} & \mathbf{H}_{33} & \mathbf{H}_{35} & \mathbf{H}_{37} \\ \mathbf{H}_{51} & \mathbf{H}_{53} & \mathbf{H}_{55} & \mathbf{H}_{57} \\ \mathbf{H}_{71} & \mathbf{H}_{73} & \mathbf{H}_{75} & \mathbf{H}_{77} \end{pmatrix} \qquad \mathbf{K} = \begin{pmatrix} \mathbf{K}_{22} & \mathbf{K}_{24} & \mathbf{K}_{26} \\ \mathbf{K}_{42} & \mathbf{K}_{44} & \mathbf{K}_{46} \\ \mathbf{K}_{62} & \mathbf{K}_{64} & \mathbf{K}_{66} \end{pmatrix}. \tag{137}$$

Since $\mathbf{Q}(t)$ is a Laurent matrix, it follows that $\mathbf{H}$ and $\mathbf{K}$ must themselves be Laurent matrices. However, by hypothesis, we have that

$$\widetilde{\mathbf{S}(t)}\mathbf{G}\mathbf{S}(t) = \alpha t^\beta \mathbf{g} \tag{138}$$

which means that

$$\widetilde{\mathbf{Q}(t)}\widetilde{\mathbf{R}(t)}\mathbf{g}\mathbf{R}(t)\mathbf{Q}(t) = \alpha t^\beta \tag{139}$$

so that, taking determinants of both sides of the above leads to the necessary condition that

$$(\det \mathbf{Q}(t))^2(\det \mathbf{R}(t))^2 = (\alpha t^\beta)^{2\ell+1} \tag{140}$$

which implies that β must be even. (Otherwise, the left-hand side of (140) would be a multiple of an even power of t and the right-hand side would be a multiple of an odd power of t.) Substituting further,

$$\widetilde{\mathbf{Q}(t)}\mathrm{diag}\{1,\ldots,1,t^{-1},t^{-1},1,\ldots,1\}\mathbf{Q}(t)=\alpha t^{\beta}\mathbf{g} \tag{141}$$

which implies in turn, using the general form for $\mathbf{Q}(t)$ given in (135), that

$$\tilde{\mathbf{H}}\mathrm{diag}\{1,\ldots,1,t^{-1},1,\ldots,1\}\mathbf{H}=\alpha t^{\beta}\mathrm{offdiag}\{1,\ldots,1\} \tag{142}$$

and upon taking determinants of (141), we find the required contradiction, namely that the left-hand side is a multiple of an odd power of t, whilst the right-hand side is a multiple of an even power of t. This concludes the work of this section, and together with the conjugacy classes $(A)^{(0)},\ldots,(A)^{(\ell)}$ identified earlier, we have thus identified conjugacy classes

$$(B)^{(0)},\ldots,(B)^{(\ell-1)}$$

with $(B)^{(m)}$ being defined to contain the automorphism $\{\mathbf{U}_m(t),1,\xi\}$.

6. Study of the conjugacy classes of the type 1a involutive automorphisms with $u=-1$

First we recall that each such automorphism is conjugate to an automorphism $\{\mathbf{U}(t),-1,\xi\}$, where $\mathbf{U}(t)$ is given by one of (51)–(58). In fact, we will show in this section that *all* of the type 1a involutive automorphisms with $u=-1$ belong to the *same* conjugacy class, which corresponds to $\{\mathbf{1}_{2\ell+1},1,0\}$. We shall call this conjugacy class (C).

The first stage is to employ the results of section 4 to show that each type 1a involutive automorphism with $u=-1$ is conjugate to a type 1a automorphism $\{\mathbf{U},-1,0\}$, where $\mathbf{U}$ is t-independent, and is given by (51). This dramatically reduces the number of automorphisms that have to be considered. We shall now complete this process fully, by showing that all of the automorphisms $\{\mathbf{D}_\ell^\circ,-1,0\}$ do, in fact, belong to the same class.

Let the quantities μ_m be defined by

$$\mu_m=\begin{cases}0 & \lambda_m=1\\ 1 & \lambda_m=-1\end{cases} \tag{143}$$

and define a matrix $\mathbf{S}(t)$ by

$$\mathbf{S}(t)=\mathrm{diag}\{t^{\mu_1},\ldots,t^{\mu_\ell},1,t^{-\mu_\ell},\ldots,t^{-\mu_1}\}. \tag{144}$$

Then

$$\widetilde{\mathbf{S}(t)}\mathbf{g}\mathbf{S}(t)=\mathbf{g} \tag{145}$$

$$\mathbf{S}(t)\mathbf{1}_{2\ell+1}\mathbf{S}(-t)^{-1}=\mathbf{D}_\ell^\circ \tag{146}$$

from which we infer that all of the automorphisms under consideration belong to just one conjugacy class.

7. Study of the conjugacy classes of the type 2a involutive automorphisms with $u = 1$

In this section we are going to investigate the automorphisms $\{\mathbf{U}(t), 1, \xi\}$, where $\mathbf{U}(t)$ is given by one of (59)–(66). We know that every type 2a involutive automorphism with $u = 1$ is conjugate to at least one of these. In fact, we can restrict our analysis considerably, by using the results of section 4, as we did for the previous section. As in the previous section, it follows that every type 2a involutive automorphism with $u = 1$ is conjugate to one of the type 2a involutive automorphisms $\{\mathbf{U}(t), 1, \xi\}$, where $\mathbf{U}(t)$ is given by (66).

The similarity with section 6 ends here though, since these automorphisms do *not* all belong to the same conjugacy class. Instead we look at the automorphisms $\{\mathbf{U}(t), 1, \xi\}$. Here $\mathbf{U}(t)$ is given by

$$\mathbf{U}(t) = \mathrm{dsum}\{t\mathbf{1}_{n_+^t}, 1_{n_+}, -\mathbf{1}_{n_-}, -t\mathbf{1}_{n_-^t}, 1, -t^{-1}\mathbf{1}_{n_-^t}, -\mathbf{1}_{n_-}, \mathbf{1}_{n_+}, t^{-1}\mathbf{1}_{n_+^t}\} \tag{147}$$

where the quantities n_+ etc are dependent upon the matrix involved, and are integers that relate to the various numbers of diagonal entries of $\mathbf{U}(t)$ that take the values 1, -1, t and t^{-1}. For example, if $\mathbf{U}(t)$ is such that it contains $2b+1$ diagonal entries $+1$ then we define n_+ in this case to be b. Clearly,

$$n_+ + n_+^t + n_- + n_-^t = \ell\,. \tag{148}$$

If we look at two matrices $\mathbf{U}_1(t)$ and $\mathbf{U}_2(t)$ then to distinguish between the values of n_+ relating to these matrices we use the expressions $n_+[\mathbf{U}_1(t)]$ and $n_+[\mathbf{U}_2(t)]$.

Section 4 demonstrates that each of the automorphisms under consideration are conjugate to one of the above. The aim of this section is to show that there are precisely $\frac{1}{2}(\ell+1)(\ell+2)$ conjugacy classes of this type, and we shall do this in a number of stages. To begin with, we shall show that two automorphisms (with associated matrices $\mathbf{U}_1(t)$ and $\mathbf{U}_2(t)$ of the form given in (147) are conjugate if both of the following hold:

$$n_+(\mathbf{U}_1) + n_+^t(\mathbf{U}_1) = n_+(\mathbf{U}_2) + n_+^t(\mathbf{U}_2) \qquad n_+^t(\mathbf{U}_1) - n_-^t(\mathbf{U}_1) = n_+^t(\mathbf{U}_2) - n_-^t(\mathbf{U}_2)\,. \tag{149}$$

To demonstrate that this is true, we need only show that it is possible to increase both n_+^t and n_-^t by an arbitrary number, and it suffices to show that they can both be increased by 1. Thus, if $\mathbf{U}(t)$ is given by (147) and we define $\mathbf{S}(t)$ by

$$\mathbf{S}(t) = \mathrm{dsum}\left\{\mathbf{1}_p, \frac{1}{2}\begin{pmatrix}(1+t) & (1-t)\\(1-t) & (1+t)\end{pmatrix}\mathbf{1}_q, \frac{1}{2}\begin{pmatrix}(1+t^{-1}) & (t^{-1}-1)\\(t^{-1}-1) & (1+t^{-1})\end{pmatrix}\mathbf{1}_p\right\} \tag{150}$$

where $p = (n_+^t + n_+ - 1)$ and $2p + 3 + q = 2\ell$, then

$$\widetilde{\mathbf{S}(t)}\mathbf{g}\mathbf{S}(t) = \mathbf{g} \tag{151}$$

and also

$$\mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(t^{-1})^{-1} = \mathbf{U}'(t) \tag{152}$$

where $\mathbf{U}'(t)$ is obtained from $\mathbf{U}(t)$ by setting the $(x-1)$th diagonal entry to t, the xth diagonal entry to $-t$, the $(2\ell+3-x)$th diagonal entry to t^{-1} and the $(2\ell+2-x)$th diagonal entry to $-t^{-1}$. Then, by a simple re-ordering of the index set, we have a matrix of the form given by (147), but such that the quantities n_+^t and n_-^t have both been increased by 1.

Moreover, for a given matrix $\mathbf{U}(t)$, define quantities N_+ and N_- by

$$N_+ = n'_+ + n_+ \qquad N_- = n'_- + n_- . \tag{153}$$

Thus, for a given $\mathbf{U}(t)$, it follows that the quantity $(n'_+ - n'_-)$ can take any one of the values $-N_-, -N_- + 1, \ldots, 0, \ldots, N_+ - 1, N_+$. This means that $(n'_+ - n'_-)$ can take any one of $(\ell + 1)$ distinct values. The next step in our analysis is to show that we can discount those automorphisms $\{\mathbf{U}(t), 1, \xi\}$ for which $(n'_+ - n'_-)$ is less than zero.

Consider such an automorphism, for which $\mathbf{U}(t)$ may be assumed to be given by

$$\mathbf{U}(t) = \mathrm{dsum}\{\mathbf{1}_j, -\mathbf{1}_k, -t\mathbf{1}_m, 1, -t^{-1}\mathbf{1}_m, -\mathbf{1}_k, \mathbf{1}_j\} . \tag{154}$$

Consider $\mathbf{U}(st)$, where s may take values 1 or -1. If we put $s = -1$ in the matrix $\mathbf{U}(t)$ then, by a suitable re-ordering of the index set (as in section 4), we may infer that $\{\mathbf{U}(t), 1, \xi\}$ is conjugate to some automorphism $\{\mathbf{V}(t), 1, \xi\}$, where

$$N_+[\mathbf{V}(t)] = j + m \qquad n'_+[\mathbf{V}(t)] - n'_-[\mathbf{V}(t)] = m . \tag{155}$$

We have therefore reduced the original set of automorphisms to a set of $(\ell + 1)(\ell + 2)/2$ automorphisms. We define matrices $\mathbf{U}_{a,b}(t)$ to be such that

$$N_+[\mathbf{U}_{a,b}(t)] = a \qquad n'_+[\mathbf{U}_{a,b}(t)] = b \qquad n'_-[\mathbf{U}_{a,b}(t)] = 0 . \tag{156}$$

Thus, we have only to consider the automorphisms $\{\mathbf{U}_{a,b}(t), 1, \xi\}$, where a runs over all integer values from 0 to ℓ, and b runs over all integer values from 0 to a. From what we have seen, it follows that every type 2a automorphism with $u = 1$ is conjugate to one of these. It now remains for us to prove that these automorphisms are all disjoint.

Suppose that there exists some $\mathbf{S}(t)$ such that the following holds:

$$\mathbf{S}(t)\mathbf{U}_{a,c}(st^{\pm})\mathbf{S}(t^{-1})^{-1} = \lambda t^{\mu}\mathbf{U}_{a,d}(t) \tag{157}$$

where we are assuming that $c \neq d$. Now there are, of course, two possibilities in the above, namely $s = 1$ and $s = -1$. Consider the first, and let t take the value -1. The equation (157) may then be satisfied only if $c = d$, which is obviously a contradiction, so we must have $s = -1$. In this case, by putting $t = -1$ we obtain the necessary condition that $c = d$, which gives the same contradiction.

Finally, we have to consider the possibility that

$$\mathbf{S}(t)\mathbf{U}_{a,b}(st^{\pm})\mathbf{S}(t^{-1})^{-1} = \lambda t^{\mu}\mathbf{U}_{c,d}(t) \tag{158}$$

where we assume only that $a \neq c$. If $s = 1$, then putting $t = 1$ in (158) implies that a must equal c, which gives a contradiction. The only remaining possibility is that $s = -1$. However, with $s = -1$ and $t = 1$ in (158), a necessary condition is that $a = c - d$, and with $s = -1$ and $t = -1$, a necessary condition is that $c = a - b$. Since (158) must hold for all values of t, it must certainly hold for $t = 1$ and -1 and so both of the following hold:

$$a = c - d \qquad c = a - b . \tag{159}$$

The first of these implies that $a \leqslant c$, and the second that $c \leqslant a$. Thus $a = c$, which again gives a contradiction.

Let $(D)^{(a,b)}$ be the conjugacy class that contains the automorphism

$$\{\mathbf{U}_{a,b}(t), 1, \xi\}$$

where a takes all integer values from 0 to ℓ, and b takes values from 0 to a, for each permitted value of a.

8. Summary of results

We conclude by giving an explicit representative for each of the conjugacy classes that we identified in the previous sections. To begin with, we note that the conjugacy classes may be considered as four 'series', namely

(1) The conjugacy classes $(A)^{(0)}, \ldots, (A)^{(\ell)}$.
(2) The conjugacy classes $(B)^{(0)}, \ldots, (B)^{(\ell-1)}$.
(3) The conjugacy class (C).
(4) The conjugacy classes $(D)^{(a,b)}$, where $0 \leqslant a \leqslant \ell$ and $0 \leqslant b \leqslant a$.

Consider first the conjugacy class $(A)^{(j)}$, where j takes an integer values from 0 to ℓ inclusive. We take as a representative of this the automorphism ϕ which is specified by the following:

$$\begin{aligned}
&\phi(h_{\alpha_k}) = h_{\alpha_k} \quad k = 1, 2, \ldots, \ell \qquad \phi(e_{\pm\alpha_0}) = \begin{cases} e_{\pm\alpha_0} & j = 0, \ldots, \ell-1 \\ -e_{\pm\alpha_0} & j = \ell \end{cases} \\
&\phi(e_{\pm\alpha_k}) = \begin{cases} e_{\pm\alpha_k} & k \neq \ell - j; k \neq \ell \\ -e_{\pm\alpha_k} & k = \ell - j; k \neq \ell \end{cases} \\
&\phi(e_{\pm\alpha_\ell}) = \begin{cases} e_{\pm\alpha_\ell} & j = 0 \\ -e_{\pm\alpha_\ell} & j = 1, 2, \ldots, \ell \end{cases} \\
&\phi(c) = c \qquad \phi(d) = d\,.
\end{aligned} \tag{160}$$

For the conjugacy classes $(B)^{(j)}$ we take the following automorphisms ϕ as representatives:

$$\begin{aligned}
&\phi(h_{\alpha_k}) = h_{\alpha_k} \quad k = 1, 2, \ell-2 \qquad \phi(h_{\alpha_{\ell-1}}) = h_{\alpha_{\ell-1}} + 2h_{\alpha_\ell} + c \\
&\phi(h_{\alpha_\ell}) = -h_{\alpha_\ell} - c \qquad \phi(e_{\pm\alpha_0}) = e_{\pm\alpha_0} \\
&\phi(e_{\pm\alpha_k}) = \begin{cases} e_{\pm\alpha_k} & k \neq j; k = 1, \ldots, \ell-2 \\ -e_{\pm\alpha_k} & k = j; k = 1, \ldots, \ell-2 \end{cases} \\
&\phi(e_{\pm\alpha_{\ell-1}}) = \begin{cases} -e_{\pm(\delta+\alpha_{\ell-1}+2\alpha_\ell)} & j = 0, 1, \ldots, \ell-2 \\ e_{\pm(\delta+\alpha_{\ell-1}+2\alpha_\ell)} & j = \ell-1 \end{cases} \\
&\phi(e_{\pm\alpha_\ell}) = -e_{\mp(\delta+\alpha_\ell)} \qquad \phi(c) = c \qquad \phi(d) = -2(2\ell-1)h_{\alpha_\ell} + (2\ell-1)c + d\,.
\end{aligned} \tag{161}$$

For the conjugacy class (C), we take as the representative the automorphism ϕ specified by:

$$\begin{aligned}
&\phi(h_{\alpha_k}) = h_{\alpha_k} \quad k = 1, 2, \ldots, \ell \qquad \phi(e_{\pm\alpha_k}) = e_{\pm\alpha_k} \quad k = 1, 2, \ldots, \ell \\
&\phi(e_{\pm\alpha_0}) = -e_{\pm(\delta+\alpha_0)} \qquad \phi(c) = c \qquad \phi(d) = d\,.
\end{aligned} \tag{162}$$

Finally, a representative ϕ for $(D)^{(a,b)}$ is defined by

$$\phi(h_{\alpha_k}) = \begin{cases} h_{\alpha_k} & k \neq b \\ h_{\alpha_b} + c & k = b \end{cases}$$

$$\phi(e_{\pm\alpha_k}) = \begin{cases} -e_{\pm\alpha_k} & k = a \neq b; k = 1, \ldots, \ell - 1 \\ -e_{\pm(\delta+\alpha_k)} & k = a = b; k = 1, \ldots, \ell - 1 \\ e_{\pm(\delta+\alpha_k)} & k = b \neq a; k = 1, \ldots, \ell - 1 \\ e_{\pm\alpha_k} & k \neq a, k \neq b; k = 1, \ldots, \ell - 1 \end{cases} \tag{163}$$

$$\phi(e_{\pm\alpha_\ell}) = \begin{cases} e_{\pm\alpha_\ell} & k = \ell \\ -e_{\pm\alpha_\ell} & k \neq \ell \end{cases} \qquad \phi(e_{\pm\alpha_0}) = \begin{cases} -e_{\mp(2\delta+\alpha_0)} & a = b = 0 \\ e_{\mp(2\delta+\alpha_0)} & a > 0; n = 0 \\ e_{\mp(3\delta+\alpha_0)} & a > 0; b > 0 \end{cases}$$

$$\phi(c) = -c \qquad \phi(d) = 2(2\ell - 1)\sum_{p=1}^{b} p h_{\alpha_p} + b \sum_{p=b+1}^{\ell} h_{\alpha_p} - (2\ell - 1)bc - d\,.$$

The Lie algebra B_1 is isomorphic to the Lie algebra A_1. Thus the results of this paper for the special case $\ell = 1$ should agree with the results of Cornwell [2]. A brief inspection reveals that this is so.

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